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LETTER TO THE EDITOR

Anisotropic scaling in layered aperiodic Ising systems

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Abstract. The influence of a layered aperiodic modulation of the couplings on the critical behaviour of the two-dimensional Ising model is studied in the case of marginal perturbations. The aperiodicity is found to induce anisotropic scaling. The anisotropy exponent z, given by the sum of the surface magnetization scaling dimensions, depends continuously on the modulation amplitude. Thus these systems are scale invariant but not conformally invariant at the critical point.

The critical behaviour of quasiperiodic or aperiodic systems is better understood since Luck recently proposed a relevance–irrelevance criterion [1]. As in the Harris criterion for random systems [2], the strength of the fluctuations of the couplings, on a scale given by the correlation length, is of primary importance for the critical behaviour. Thus an aperiodic perturbation can be relevant, marginal or irrelevant, depending on the sign of a crossover exponent involving the correlation length exponent of the unperturbed system ν and the wandering exponent ω which governs the fluctuations of the aperiodic sequence [3]. The criterion explains earlier results (for references see [1]) and has been confirmed in recent work on the layered 2D Ising model [4–6].

In this letter, we report on some recent results supporting the occurrence of anisotropic scaling in the two-dimensional layered Ising model with a marginal aperiodic modulation of the exchange interactions.

We consider a system with a constant interaction K_1 along the layers and aperiodically modulated interactions $K_2(k)$ (in k_BT units) between neighbouring layers at k and $k+1$. In the extreme anisotropic limit, $K_1 \to \infty$, $K_2(k) \to 0$, the row-to-row transfer operator involves the Hamiltonian of a quantum Ising chain [7]

$$\mathcal{H} = -\tfrac{1}{2}\sum_k \left[\sigma_k^z + \lambda_k \sigma_k^x \sigma_{k+1}^x\right] \tag{1}$$

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where the σs are Pauli spin operators and the coupling λ_k is given by the ratio $-2K_2(k)/\ln(\tanh K_1)$. For the aperiodic system, we use the parametrization $\lambda_k = \lambda r^{f_k}$ where f_k takes the values 0 or 1 given by an aperiodic sequence which is constructed through substitution.

In the following, we shall consider:

- the *period-doubling sequence* [8] with the substitutions $\mathcal{S}(1) = 1\ 0$, $\mathcal{S}(0) = 1\ 1$, so that, after n iterations, one obtains

$$\begin{array}{ll} n=0 & 1 \\ n=1 & 1\,0 \\ n=2 & 1\,0\,1\,1 \\ n=3 & 1\,0\,1\,1\,1\,0\,1\,0 \end{array} \tag{2}$$

- the *paper-folding sequence* [9] with the two-digit substitutions $\mathcal{S}(1\ 1) = 1\ 1\ 0\ 1$, $\mathcal{S}(1\ 0) = 1\ 1\ 0\ 0$, $\mathcal{S}(0\ 1) = 1\ 0\ 0\ 1$, $\mathcal{S}(0\ 0) = 1\ 0\ 0\ 0$, leading to

$$\begin{array}{ll} n=0 & 1\,1 \\ n=1 & 1\,1\,0\,1 \\ n=2 & 1\,1\,0\,1\,1\,0\,0\,1 \\ n=3 & 1\,1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0\,1\,0\,0\,1 \end{array} \tag{3}$$

- the *three-folding sequence* [10] which follows from the substitutions $\mathcal{S}(0) = 0\ 1\ 0$, $\mathcal{S}(1) = 0\ 1\ 1$, giving

$$\begin{array}{ll} n=0 & 0 \\ n=1 & 0\,1\,0 \\ n=2 & 0\,1\,0\,0\,1\,1\,0\,1\,0 \\ n=3 & 0\,1\,0\,0\,1\,1\,0\,1\,0\,0\,1\,0\,0\,1\,1\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1\,0. \end{array} \tag{4}$$

Most of the properties of a sequence can be deduced from its substitution matrix [11] with entries M_{ij} given by the numbers of digits (or pairs) of the different types in the substitutions.

On a chain with length L, the fluctuations of the couplings can be measured through the cumulated deviation from their average $\overline{\lambda}$, which behaves as [3]

$$\Delta(L) = \sum_{k=1}^{L}(\lambda_k - \overline{\lambda}) \approx \delta L^{\omega} F\left(\frac{\ln L}{\ln \Lambda_1}\right). \tag{5}$$

Here $\delta = \lambda(r-1)$ is the amplitude of the modulation. The wandering exponent ω is given by the ratio

$$\omega = \frac{\ln|\Lambda_2|}{\ln \Lambda_1} \tag{6}$$

where Λ_1 is the largest and Λ_2 the next-to-largest eigenvalue of the substitution matrix. $F(x)$ is a fractal periodic function of its argument with period unity [3].

Under a change of the length scale by $b = L/L'$, the amplitude δ is changed into [4]

$$\delta' = b^{\omega-1+1/\nu}\,\delta \tag{7}$$

where ν is the correlation length exponent. This leads to the Luck criterion [1, 12, 13] *according to which the aperiodic perturbation is relevant (irrelevant) when* $\omega > (<)\, 1 - 1/\nu$. On the border, where $\omega = 1 - 1/\nu$, the perturbation is marginal and leads to δ-dependent exponents. For the 2D Ising model with $\nu = 1$, marginal behaviour is expected for $\omega = 0$ which is the value of the wandering exponent for the three sequences mentioned above.

The surface magnetization of the Ising quantum chain takes the simple form [14]

$$m_s = \left(1 + \sum_{j=1}^{\infty} \prod_{k=1}^{j} \lambda_k^{-2}\right)^{-1/2} \underset{\lambda\to\lambda_c+}{\sim} t^{\beta_s} \qquad t = 1 - (\lambda_c/\lambda)^2 . \tag{8}$$

Here $\lambda_c = r^{-\rho_\infty}$ is the critical coupling [15, 4] and ρ_∞ is the asymptotic density of the digit 1 along the sequence. The surface magnetization exponent $\beta_s = x_{ms}$, where x_{ms} is the scaling *dimension of the surface spins, since* $\nu = 1$. *The surface magnetization can generally be* evaluated recursively [4] and the critical exponent is obtained using a finite-size-scaling method [16]. This has been done for the period-doubling sequence for which [4]

$$x_{ms} = \overline{x_{ms}} = \frac{\ln\left(\lambda_c^{1/2} + \lambda_c^{-1/2}\right)}{2\ln 2} \qquad \lambda_c = r^{-2/3} \qquad \text{(period-doubling)} \tag{9}$$

where $\overline{x_{ms}}$ is the exponent on the surface corresponding to the right-hand end of the sequence. The period-doubling sequence in (2) is symmetric if one ignores the last digit which does not influence the critical behaviour. As a consequence $x_{ms} = \overline{x_{ms}}$, i.e. the surface exponents are the same on both sides.

Similar calculations for the two last sequences give the following results [17]:

$$x_{ms} = \frac{\ln\left(1 + \lambda_c^2\right)}{2\ln 2} \qquad \overline{x_{ms}} = \frac{\ln\left(1 + \lambda_c^{-2}\right)}{2\ln 2} \qquad \lambda_c = r^{-1/2} \qquad \text{(paper-folding)} \tag{10}$$

$$x_{ms} = \frac{\ln\left(2 + \lambda_c^{-2}\right)}{2\ln 3} \qquad \overline{x_{ms}} = \frac{\ln\left(2 + \lambda_c^{2}\right)}{2\ln 3} \qquad \lambda_c = r^{-1/2} \qquad \text{(three-folding)} \tag{11}$$

where the values for the right-hand surface are obtained by changing r into r^{-1} since, except for the last digit, viewed from the right-hand side the sequences in (3) and (4) are obtained by exchanging 0 and 1. Similar expressions, involving K_1 and K_2, are obtained on the corresponding 2D classical systems [18].

The excitation spectrum of the Hamiltonian (1) has been studied numerically for the three marginal sequences given above†. A quadratic fermion Hamiltonian is first obtained via the Jordan–Wigner transformation [22], which is diagonalized using standard methods [23]. At the critical point, the low-lying fermion excitations ϵ_n are found to scale with the size of the system L as

$$\epsilon_n \sim L^{-z} \qquad z = x_{ms} + \overline{x_{ms}} . \tag{12}$$

† Although the structure of the spectra of aperiodic Ising quantum chains has already been studied [19–21, 1], either the aperiodicity was an irrelevant perturbation or the scaling behaviour was not discussed.

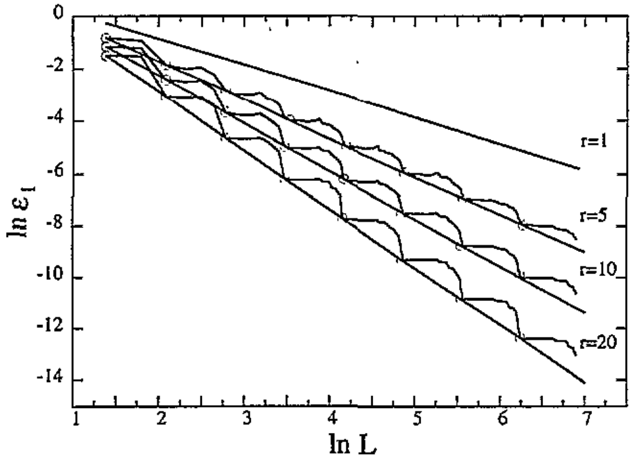

Figure 1. Log–log plots of the first excitation energy ϵ_1 versus the length L of the chain for the paper-folding sequence with periodic boundary conditions and different values of r. The slopes $-z$ are taken for values of L equal to 2^n (○), which correspond to a constant amplitude.

Table 1. Extrapolated finite-size estimates for the exponent z, obtained from the fermion excitations ϵ_n for the period-doubling sequence. The figures in brackets give the estimated uncertainty in the last digit.

r	ϵ_1	ϵ_2	ϵ_3	ϵ_4	ϵ_5	ϵ_6	Expected
5.0	1.198 34(5)	1.198 36(1)	1.198 357(1)	1.198 36(1)	1.198 36(1)	1.198 355(5)	1.198 356
	1.198 36(2)	1.198 35(1)	1.198 35(2)	1.198 357(5)	1.198 359(4)	1.198 364(4)	
4.0	1.148 85(1)	1.148 84(2)	1.148 842(4)	1.148 845(2)	1.148 84(1)	1.148 844(4)	1.148 844
	1.148 84(1)	1.148 84(1)	1.148 842(3)	1.148 84(1)	1.148 844(3)	1.148 84(1)	
3.0	1.094 65(1)	1.094 648(4)	1.094 649(4)	1.094 647(3)	1.094 65(1)	1.094 651(2)	1.094 649
	1.094 65(1)	1.094 654(2)	1.094 653(3)	1.094 647(3)	1.094 648(4)	1.094 647(3)	
2.0	1.038 17(5)	1.038 17(2)	1.038 20(5)	1.003 817(4)	1.038 17(4)	1.038 17(1)	1.038 170
	1.038 1(1)	1.038 17(1)	1.038 172(6)	1.038 174(5)	1.038 172(3)	1.038 171(2)	
0.5	1.038 1(2)	1.038 14(5)	1.038 15(5)	1.038 15(5)	1.038 16(2)	1.038 16(1)	1.038 170
	1.038 1(1)	1.038 16(3)	1.038 17(3)	1.0381(1)	1.0381(1)	1.038 17(1)	

The same behaviour is obtained for free and periodic boundary conditions [17, 18, 24].

The oscillations around the power laws, as shown for the paper-folding sequence on figure 1, are due to a periodic prefactor which, like $F(x)$ in (5), is a function of $\ln L/\ln\Lambda_1$. Oscillating amplitudes are obtained for other critical quantities as well. When the size of the system goes to infinity, L is replaced by the correlation length $\xi \sim t^{-1}$ near the critical point, and the argument of the fractal function involves the ratio $\ln t/\ln\Lambda_1$.

In table 1 we give finite-size estimates for z, supporting (9) and (12), which were obtained from sequence extrapolation using the BST algorithm (see [25]). Chains of size $L = 2^n + 1$ up to $n = 20$ with free boundary conditions were used. For a given r, the first line in table 1 refers to data with n even, while the second line refers to n odd. Even and odd values of n give a different amplitude since the period of the fractal function is 2 in this case.

The behaviour of the excitation spectrum in (12) is typical of a strongly anisotropic system [26] with a correlation length exponent $\nu_\parallel = z\nu$ in the time direction (i.e. along the layers). In the transverse direction, the correlation length exponent keeps its unperturbed value $\nu = 1$ since otherwise the perturbation would not remain marginal. Although

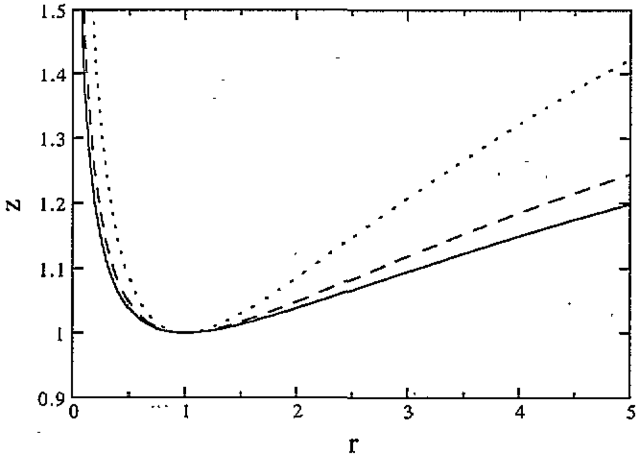

Figure 2. Anisotropy exponent z as a function of the strength r of the aperiodic modulation for the period-doubling (full curve), three-folding (broken curve) and paper-folding (dotted curve) sequences.

anisotropic critical behaviour is rather common, what is remarkable here is the occurrence of a continuously varying anisotropy $z = z(r)$ (see figure 2). The anisotropy is also implicit in Luck's finite-size calculation of an effective sound velocity [1].

The singular part of the bulk energy density is found numerically to scale like L^{-z} at the critical point so that its scaling dimension is given by

$$x_e = x_{\rm ms} + \overline{x_{\rm ms}} \tag{13}$$

in agreement with anisotropic scaling for the bulk free energy density [26]

$$f(t, L) = b^{-(1+z)} f(b^{1/\nu} t, L/b) \tag{14}$$

where t is the deviation from the critical coupling as defined in (8). From (14), the specific heat exponent is given by

$$\alpha = 1 - z = 1 - x_{\rm ms} - \overline{x_{\rm ms}} \tag{15}$$

a relation which is indeed verified numerically for the period-doubling sequence. It takes a negative value since the surface magnetization exponents are always greater than $\frac{1}{2}$, the pure system value: the aperiodic modulation of the coupling weakens the critical singularities. One has to notice that the proposed analytical expression for the specific heat exponent of the period-doubling sequence, which is obtained by combining (9) and (15), leads to $\alpha \approx -\Delta^2/\ln 2$ for a weak perturbation where $\Delta^2 = \rho_\infty(1 - \rho_\infty)(\ln r)^2$ in our notation. This disagrees with Luck's numerical results, which involved a supplementary scaling assumption [1].

On a semi-infinite system, using a finite-size scaling argument [17], the scaling dimensions $x_{\rm es}$ of the surface energy density on the left surface can be related to the dynamical exponent z and the corresponding surface magnetization exponent. It is given by

$$x_{\rm es} = z + 2x_{\rm ms} \tag{16}$$

which agrees with the numerical results. A similar expression is obtained on the right surface.

According to anisotropic scaling [26], the critical spin–spin correlation function on the left surface is expected to behave as

$$G_s(r_\parallel, t) = b^{-2x_{ms}} G_s(r_\parallel / b^z, b^{1/\nu} t) . \tag{17}$$

At the critical point $t = 0$, the decay exponent is given by $2x_{ms}/z = 2x_{ms}/(x_{ms} + \overline{x_{ms}})$ instead of $2x_{ms}$ for an isotropic system. Such a modified decay has been obtained on the 2D classical system [18] as well as for the quantum chain [17]. When the sequence is symmetric the decay exponent is equal to 1, i.e. it is the same as for the unperturbed system. Such a behaviour is indeed obtained with the period-doubling sequence.

Marginal aperiodic perturbations of the Ising quantum chain have been shown to induce strongly anisotropic critical behaviour. The anisotropy exponent z, which is found numerically to be the sum of the surface magnetization scaling dimensions x_{ms} and $\overline{x_{ms}}$, varies continuously with the amplitude of the aperiodicity. The values of other bulk and surface exponents have been conjectured on the basis of numerical results and scaling assumptions. The bulk magnetic behaviour remains to be studied. Details will be given in forthcoming publications [17, 18, 24]

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